

# HÖLDER CONTINUITY OF HARMONIC QUASICONFORMAL MAPPINGS★

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**ABSTRACT.** We prove that for harmonic quasiconformal mappings  $\alpha$ -Hölder continuity on the boundary implies  $\alpha$ -Hölder continuity of the map itself. Our result holds for the class of uniformly perfect bounded domains, in fact we can allow that a portion of the boundary is thin in the sense of capacity. The problem for general bounded domains remains open.

**Keywords.** Quasiconformal maps, harmonic mappings, Hölder continuity.

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## 1. INTRODUCTION

The following theorem is the main result in [8].

**Theorem 1.1.** *Let  $D$  be a bounded domain in  $\mathbb{R}^n$  and let  $f$  be a continuous mapping of  $\overline{D}$  into  $\mathbb{R}^n$  which is quasiconformal in  $D$ . Suppose that, for some  $M > 0$  and  $0 < \alpha \leq 1$ ,*

$$(1.2) \quad |f(x) - f(y)| \leq M|x - y|^\alpha$$

*whenever  $x$  and  $y$  lie on  $\partial D$ . Then*

$$(1.3) \quad |f(x) - f(y)| \leq M'|x - y|^\beta$$

*for all  $x$  and  $y$  on  $\overline{D}$ , where  $\beta = \min(\alpha, K_I^{-1/(1-n)})$  and  $M'$  depends only on  $M$ ,  $\alpha$ ,  $n$ ,  $K(f)$  and  $\text{diam}(D)$ .*

The exponent  $\beta$  is the best possible, as the example of a radial quasiconformal map  $f(x) = |x|^{\alpha-1}x$ ,  $0 < \alpha < 1$ , of  $\overline{\mathbb{B}^n}$  onto itself shows (see [11], p. 49). Also, the assumption of boundedness is essential. Indeed, one can consider  $g(x) = |x|^a x$ ,  $|x| \geq 1$  where  $a > 0$ . Then  $g$  is quasiconformal in  $D = \mathbb{R}^n \setminus \overline{\mathbb{B}^n}$  (see [11], p. 49), it is identity on  $\partial D$  and hence Lipschitz continuous on  $\partial D$ . However,  $|g(te_1) - g(e_1)| \asymp t^{a+1}$ ,  $t \rightarrow \infty$ , and therefore  $g$  is not globally Lipschitz continuous on  $D$ .

This paper deals with the following question, suggested by P. Koskela: is it possible to replace  $\beta$  with  $\alpha$  if we assume, in addition to quasiconformality, that  $f$  is harmonic? In the special case  $D = \mathbb{B}^n$  this was proved, for arbitrary moduli of continuity  $\omega(\delta)$ , in [2]. Our main result is that the answer is positive, if  $\partial D$  is a uniformly perfect set (cf. [6]). In fact, we prove a more general result, including domains having a thin, in the sense of capacity, portion of the boundary. However, this generality is in a sense illusory, because any harmonic and quasiconformal (briefly hqc) mapping extends harmonically and quasiconformally across such portion of the boundary. Nevertheless, it leads to a natural open question: is the answer positive for arbitrary bounded domain in  $\mathbb{R}^n$ ?

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In the case of smooth boundaries much better regularity up to the boundary can be deduced, see [7]; related results for harmonic functions were obtained by [1].

We denote by  $B(x, r)$  and  $S(x, r)$  the open ball, respectively sphere, in  $\mathbb{R}^n$  with center  $x$  and radius  $r > 0$ . We adopt the basic notation, terminology and definitions related to quasiconformal maps from [11]. A condenser is a pair  $(K, U)$ , where  $K$  is a non-empty compact subset of an open set  $U \subset \mathbb{R}^n$ . The capacity of the condenser  $(K, U)$  is defined as

$$\text{cap}(K, U) = \inf \int_{\mathbb{R}^n} |\nabla u|^n dV,$$

where infimum is taken over all continuous real-valued  $u \in ACL^n(\mathbb{R}^n)$  such that  $u(x) = 1$  for  $x \in K$  and  $u(x) = 0$  for  $x \in \mathbb{R}^n \setminus U$ . In fact, one can replace the  $ACL^n$  condition with Lipschitz continuity in this definition. We note that, for a compact  $K \subset \mathbb{R}^n$  and open bounded sets  $U_1$  and  $U_2$  containing  $K$  we have:  $\text{cap}(K, U_1) = 0$  iff  $\text{cap}(K, U_2) = 0$ , therefore the notion of a compact set of zero capacity is well defined (see [12], Remarks 7.13) and we can write  $\text{cap}(K) = 0$  in this situation. For the notion of the modulus  $M(\Gamma)$  of a family  $\Gamma$  of curves in  $\mathbb{R}^n$  we refer to [11] and [12]. These two notions are related: by results of [5] and [13] we have

$$\text{cap}(K, U) = M(\Delta(K, \partial U; U)),$$

where  $\Delta(E, F; G)$  denotes the family of curves connecting  $E$  to  $F$  within  $G$ , see [11] or [12] for details.

In addition to this notion of capacity, related to quasiconformal mappings, we need Wiener capacity, related to harmonic functions. For a compact  $K \subset \mathbb{R}^n$ ,  $n \geq 3$ , it is defined by

$$\text{cap}_W(K) = \inf \int_{\mathbb{R}^n} |\nabla u|^2 dV,$$

where infimum is taken over all Lipschitz continuous compactly supported functions  $u$  on  $\mathbb{R}^n$  such that  $u = 1$  on  $K$ . Let us note that every compact  $K \subset \mathbb{R}^n$  which has capacity zero has Wiener capacity zero. Indeed, choose an open ball  $B_R = B(0, R) \supset K$ . Since  $n \geq 2$  we have, by Hölder inequality,

$$\int_{\mathbb{R}^n} |\nabla u|^2 dV \leq |B_R|^{1-2/n} \left( \int_{\mathbb{R}^n} |\nabla u|^n dV \right)^{2/n}$$

for any Lipschitz continuous  $u$  vanishing outside  $U$ , our claim follows immediately from definitions.

A compact set  $K \subset \mathbb{R}^n$ , consisting of at least two points, is  $\alpha$ -uniformly perfect ( $\alpha > 0$ ) if there is no ring  $R$  separating  $K$  (i.e. such that both components of  $\mathbb{R}^n \setminus R$  intersect  $K$ ) such that  $\text{mod}(R) > \alpha$ . We say that a compact  $K \subset \mathbb{R}^n$  is uniformly perfect if it is  $\alpha$ -uniformly perfect for some  $\alpha > 0$ .

We denote the  $\alpha$ -dimensional Hausdorff measure of a set  $F \subset \mathbb{R}^n$  by  $\Lambda_\alpha(F)$ .

## 2. THE MAIN RESULT

In this section  $D$  denotes a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ . Let

$$\Gamma_0 = \{x \in \partial D : \text{cap}(\overline{B}(x, \epsilon) \cap \partial D) = 0 \text{ for some } \epsilon > 0\},$$

and  $\Gamma_1 = \partial D \setminus \Gamma_0$ . Using this notation we can state our main result.

**Theorem 2.1.** *Assume  $f : \overline{D} \rightarrow \mathbb{R}^n$  is continuous on  $\overline{D}$ , harmonic and quasiconformal in  $D$ . Assume  $f$  is Hölder continuous with exponent  $\alpha$ ,  $0 < \alpha \leq 1$ , on  $\partial D$  and  $\Gamma_1$  is uniformly perfect. Then  $f$  is Hölder continuous with exponent  $\alpha$  on  $\overline{D}$ .*

If  $\Gamma_0$  is empty we obtain the following

**Corollary 2.2.** *If  $f : \overline{D} \rightarrow \mathbb{R}^n$  is continuous on  $\overline{D}$ , Hölder continuous with exponent  $\alpha$ ,  $0 < \alpha \leq 1$ , on  $\partial D$ , harmonic and quasiconformal in  $D$  and if  $\partial D$  is uniformly perfect, then  $f$  is Hölder continuous with exponent  $\alpha$  on  $\overline{D}$ .*

The first step in proving Theorem 2.1 is reduction to the case  $\Gamma_0 = \emptyset$ . In fact, we show that existence of a hqc extension of  $f$  across  $\Gamma_0$  follows from well known results. Let  $D' = D \cup \Gamma_0$ . Then  $D'$  is an open set in  $\mathbb{R}^n$ ,  $\Gamma_0$  is a closed subset of  $D'$  and  $\partial D' = \Gamma_1$ .

Clearly  $\text{cap}(K \cap \Gamma_0) = 0$  for each compact  $K \subset D'$ , and therefore, by Lemma 7.14 in [12],  $\Lambda_\alpha(K \cap \Gamma_0) = 0$  for each  $\alpha > 0$ . In particular,  $\Gamma_0$  has  $\sigma$ -finite  $(n-1)$ -dimensional Hausdorff measure. Since it is closed in  $D'$ , we can apply Theorem 35.1 in [11] to conclude that  $f$  has a quasiconformal extension  $F$  across  $\Gamma_0$  which has the same quasiconformality constant as  $f$ .

Since  $\Gamma_0$  is a countable union of compact subsets  $K_j$  of capacity zero and hence of Wiener capacity zero we conclude that  $\Gamma_0$  has Wiener capacity zero. Hence, by a classical result (see [4]), there is a (unique) extension  $G : \overline{D'} \rightarrow \mathbb{R}^n$  of  $f$  which is harmonic in  $D'$ . Obviously,  $F = G$  is a harmonic quasiconformal extension of  $f$  to  $\overline{D'}$  which has the same quasiconformality constant as  $f$ .

In effect, we reduced the proof of Theorem 2.1 to the proof of Corollary 2.2. We begin the proof of Corollary 2.2 with the following

**Lemma 2.3.** *Let  $D \subset \mathbb{R}^n$  be a bounded domain with uniformly perfect boundary. There exists a constant  $m > 0$  such that for every  $y \in D$  we have*

$$(2.4) \quad \text{cap}(\overline{B}(y, \frac{d}{2}), D) \geq m, \quad d = \text{dist}(y, \partial D).$$

*Proof.* Fix  $y \in D$  as above and  $z \in \partial D$  such that  $|y - z| = d \equiv r$ . Clearly  $\text{diam}(\partial D) = \text{diam}(D) > 2r$ . Set  $F_1 = \overline{B}(z, r) \cap (\partial D)$  and  $F_2 = \overline{B}(z, r) \cap \overline{B}(y, \frac{d}{2})$ ,  $F_3 = S(z, 2r)$ . Let  $\Gamma_{i,j} = \Delta(F_i, F_j; \mathbb{R}^n)$  for  $i, j = 1, 2, 3$ . By [6, Thm 4.1(3)] there exists a constant  $a = a(E, n) > 0$  such that

$$M(\Gamma_{1,3}) \geq a$$

while by standard estimates [11, 7.5] there exists  $b = b(n) > 0$  such that

$$M(\Gamma_{2,3}) \geq b.$$

Next, by [12, Cor 5.41] there exists  $m = m(E, n) > 0$  such that

$$M(\Gamma_{1,2}) \geq m.$$

Finally, with  $B = \overline{B}(y, d/2)$  we have

$$\text{cap}(B, D) = M(\Delta(B, \partial D; \mathbb{R}^n)) \geq M(\Gamma_{1,2}) \geq m.$$

□

In conclusion, from the above lemma, our assumption

$$|f(x_1) - f(x_2)| \leq C|x_1 - x_2|^\alpha, \quad x_1, x_2 \in \partial D,$$

and Lemma 8 in [8] we conclude that there is a constant  $M$ , depending on  $m, n, K(f), C$  and  $\alpha$  only such that

$$|f(x) - f(y)| \leq M|x - y|^\alpha, \quad y \in D, x \in \partial D, \text{dist}(y, \partial D) = |x - y|.$$

However, an argument presented in [8] shows that the above estimate holds for  $y \in D$ ,  $x \in \partial D$  without any further conditions, but with possibly different constant:

$$(2.5) \quad |f(x) - f(y)| \leq M'|x - y|^\alpha, \quad y \in D, x \in \partial D.$$

The following lemma was proved in [3] for real valued functions, but the proof relies on the maximum principle which holds also for vector valued harmonic functions, hence lemma holds for harmonic mappings as well.

**Lemma 2.6.** *Assume  $h : \overline{D} \rightarrow \mathbb{R}^n$  is continuous on  $\overline{D}$  and harmonic in  $D$ . Assume for each  $x_0 \in \partial D$  we have*

$$\sup_{B_r(x_0) \cap D'} |h(x) - h(x_0)| \leq \omega(r) \quad \text{for } 0 < r \leq r_0.$$

*Then  $|h(x) - h(y)| \leq \omega(|x - y|)$  whenever  $x, y \in D$  and  $|x - y| \leq r_0$ .*

Now we combine (2.5) and the above lemma, with  $r_0 = \text{diam}(D)$ , to complete the proof of Corollary 2.2 and therefore of Theorem 2.1 as well.

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